NASA TT F-11,474

THEORY OF THE SUPERSONIC BOOM BASED ON THE NONLINEAR THEORY OF SHORT WAVES

Jean-Pierre Guiraud

Translation of "La Theorie du bang supersonique d'apres la theorie nonlineaire des ondes courtes", Office National d'Etudes et de Recherches Aerospatiales, T.P. No. 521, pp. 1-22, October, 1967.

This paper was presented at the Franco-German Colloquium on Supersonic Booms. I.S.L., St. Louis, October 18-20, 1967.

GPO PRICE $ _____

CFSTI PRICE(S) $ _____

Hard copy (HC) 3.00

Microfiche (MF) _____

ff 653 July 65

FACILITY FORM 602

N68-30792 (ACCESSION NUMBER) (THRU)

38 (PAGES) 0 (CODE)

✓ (NASA CR OR TMX OR AD NUMBER) 02 (CATEGORY)

AUG 1968

RECEIVED
NASA STI FACILITY
INPUT BRANCH

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D.C. 20546 JUNE 1968

THEORY OF THE SUPERSONIC BOOM BASED ON THE NONLINEAR THEORY OF SHORT WAVES

J.-P. Guiraud

Faculty of Sciences ORSAY
O.N.E.R.A. Chatillon s Bagneux 92

ABSTRACT. The author investigates the origin of supersonic boom and its relation to several elementary phenomena. He shows the governing elements in the sonic perturbation and the corresponding condition of shock interpreted in terms of velocities. The nonlinearity effect is contemplated by reducing it to a phenomenon of convection along the rays of the acoustic approximation. Abundant literature on the supersonic boom is discussed.

The sonic boom owes its origin to a complex phenomenon of wave propagation and results from the competition of several /1* elementary phenomena: (I) accumulation of perturbation in the neighborhood of certain surfaces; (II) steepening of the signals by the nonlinear effect; (III) dispersion and scattering of the waves due to the dissipating effect of the viscosity and thermal conductivity.

The accumulation phenomenon is well-known by engineers in relation to the sonic boom, and in particular, it is known how to localize the favored wave surfaces (Fig. 1). What is perhaps less known is the fact that this phenomenon is commonplace from the mathematical point of view and can intervene every time time there is propagation by waves: i.e., each time certain wave surfaces can propagate discontinuities or singularities. In general, this is expressed by the fact that if a signal is "trapped," so to speak, at a certain instant in the neighborhood of one of these waves surfaces, it later accompanies this wave surface in the course of its evolution by remaining, as it were, trapped in a "wave train" whose

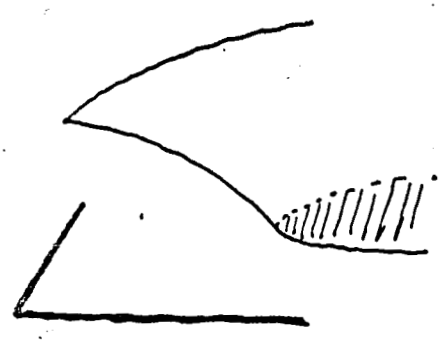


Fig. 1

* Numbers in the margin indicate pagination in the foreign text.

width in the direction of propagation is very minor compared with its transverse dimension. In addition, not only is the signal strongly trapped in a wave train, i.e., between two surfaces which move at almost the same velocity and at a short distance from each other, but it is also strongly trapped even in the ray tubes which serve as the geometrical support for the propagation phenomena. A very simple example provides an ample illustration of this fact: a sonic perturbation at instant t_1 is in the form of a divergent wave train in a spherical crown of Σ_{t_1} of very small width; later, at instant t_2, this same perturbation is found to be localized in a crown Σ_{t_2} which is deduced from the first by propagation with the speed of sound along the rays. Furthermore, the two crowns can be placed in correspondence by rays and one can recognize the same sonic perturbation in the corresponding elements.

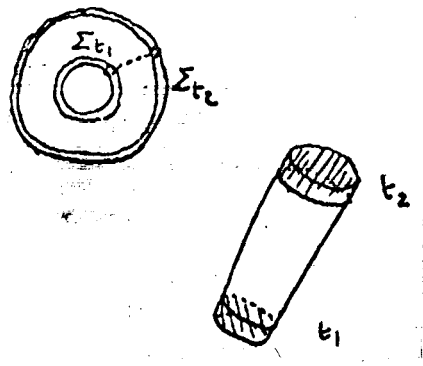


Fig. 2

The following can be stated without difficulty: the sonic perturbation is governed by the equation

$$\left(\frac{1}{a^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial R^2}\right)(ru) - \frac{1}{r^2}\delta(ru) = 0 \qquad 1$$

where the operator δ does not act on the variable r; thus, if a wave train is localized in the band $\alpha < r - at < \beta$, as indicated in Fig. 4, the derivative $\frac{1}{r^2}\delta(ru)$ becomes less and less important as the relative width of $\frac{\beta-\alpha}{r}$, becomes smaller and smaller in the course of the propagation. /2

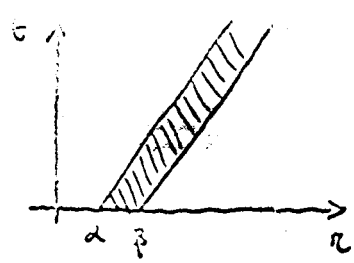
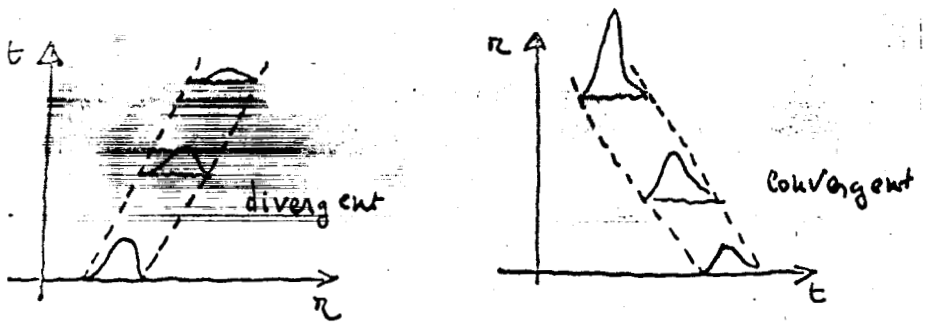


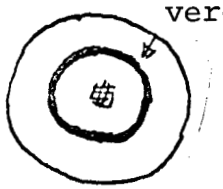
Fig. 3

This commonplace remark leads to the following result: if, because of circumstances and after a transitory phase, a wave train of small width happens to emerge successfully, the structure of this train is given approximately

$$u(\vec{x},t) \sim \frac{f(r-at,\ \vec{x}/r)}{r} \qquad r=|\vec{x}|$$

Despite its great simplicity, this formula contains the embryo of all the elements of a general theory, and for this reason its close analysis is quite instructive. Thus, r - at is a variable which at any instant describes the internal geometrical structure of the wave train; $\vec{x}/r$ is a variable, here angular in character, which differentiates the rays;





The wave is reinforced as it converges toward the center, in the vicinity of which it loses its "shortwave" structure

Fig. 4

finally, r characterizes the length travelled on a ray by following the wave, or the time elapsed. The arbitrariness of functions shows that the shape of the signal in a wave train is largely arbitrary; on the other hand, when the shape of the signal is given, at a certain instant on each ray but arbitrarily from one ray to another, this shape is later preserved on each ray with only the amplitude of the signal varying (here as I/r). For a diverging wave which spreads, the amplitude decreases, whereas for a converging wave which contracts, the amplitude increases (Fig. 5).

Fig. 5

On the basis of Fig. 2 it may be said that the cell which acts as the trap expands in the case of a diverging wave, whereas it contracts in the case of a converging wave. In the vicinity of the cen-

ter, an apparent singularity develops, but here the approximate representation (2) ceases to be valid, since the term $\frac{1}{r} = \delta(ru)$ in (I) ceases to be negligible. A completely analogous phenomenon takes place in the vicinity of the boom: the sound rays go through a caustic and the cell acting as the trap gradually fades, but long before the train has reached the caustic, the very concept of a wave train of small width loses all its meaning, and a different approach must be used. /3

All these considerations suggest a certain intellectual effort in approaching the mathematical study of these shortwave phenomena. Let F be a variable designed to describe the internal geometric structure of the wave train, F remaining constant when a particular wave is followed; let R be a designation of the rays of propagation and finally let τ be a temporal variable (r/a in the preceding example); we shall say that the phenomenon represented by u (x,t) has a shortwave structure if the following representation is approximately valid $u \sim g(\tau,R)\cdot f(F;\tau,R)$, 3 it being understood that f is zero or at least very small outside the interval , and that the area swept by the wave with τ constant in this interval is very small as compared to that which is swept with F being constant when τ varies, this being so for each R. More accurately, u is represented by the product of an amplitude function $g(\tau,R)$ independent of F and of a function of the form $f(F;\tau,R)$. The dependence of f in relation to F defines the shape of the signal, whereas the variation of g with τ defines its evolution in time. It is understood that in a cell (in the somewhat vague meaning which was given to this term above) u depends primarily on F'.

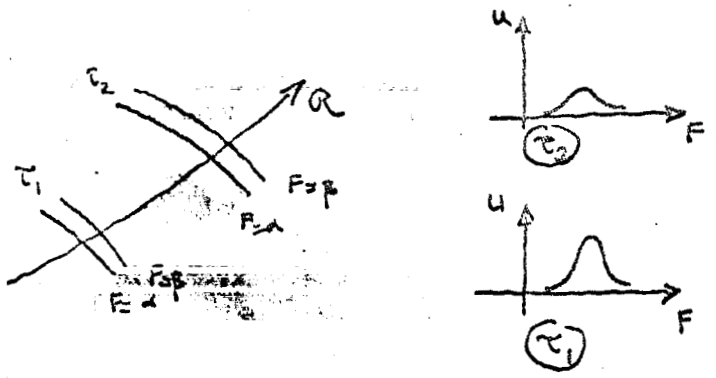


Fig. 6

A mathematical result of the greatest importance for the development and neatness of the theory is as follows: equation (I) and many equations of similar mathematical structure but more complex ones have generalized solutions or solutions in the sense of distributions, which present singularities of diverse nature on certain wave surfaces and, in particular, may present discontinuities on the surfaces in question. Thus, (2) is a generalized solution of (1) even if f presents an arbitrary discontinuity for r - at $=r_0$, depending even arbitrarily on τ. More generally, let us have a linear hyperbolic system (H) governing a phenomenon described by N functions $u_1, u_2, \ldots u_n$ (for example, com-

ponents of the velocity, pressure, volume mass in gas dynamics and in the linear approximation); with this system one can associate waves (acoustic sound waves in the example cited), i.e., surfaces moving in time and described by equations of the type F =const. The system (H) has generalized solutions $u_2,\ldots,u_n$ discontinuous on F=0, for example. If we write

$$[u_i] = (u_i)_{F=0^+} - (u_i)_{F=0^-} \qquad 4$$

the amplitude of the discontinuity of u_i, we find that /4

$$[u_i] = g(\tau, R)\, W_i(\tau, R) \qquad 5$$

The functions of structure W_i (τ, R) depend only on the equation and on the localization on the wave considered, whereas the unique function of amplitude $g(\tau, R)$ is only restricted to satisfying the evolution equation

$$\frac{\partial g}{\partial \tau} + k g = 0 \quad , \quad (k = k(\tau, R)) \qquad 6$$

We know how to form k as soon as we know the system (H) and the wave considered. Here is an example directly related to the theory of the supersonic boom: let us characterize the local state of a gas by u_1, u_2, u_3, p, ρ and let us have $u^0_1,\ldots,\rho^0$ corresponding to the state of the atmosphere which transmits the disturbance $p = u_1 - u_1^0$,...which is supposed to be of small amplitude. Let us have in M a wave surface of normal unit vector $\vec{n}$ (n_1, n_2, n_3), moving at a velocity $\vec{U}^0 + a^0 \vec{n}$ ($\vec{U}^0$ is the vector of the components u_1^0, u_2^0, u_3^0 and a^0 is the speed of sound in the atmosphere p^0, ρ^0), the theory gives

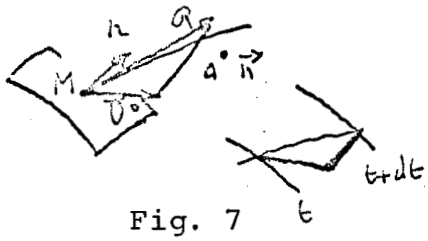


Fig. 7

$$[u_i^{(1)}] = g W_i \quad , \quad i = 1,2,3 \quad ; \quad [p^{(1)}] = g W_4 \quad ; \quad [\rho^{(1)}] = g W_5 \qquad 7$$

and we find

$$W_1 = a^0 n_1 \quad , \quad W_2 = a^0 n_2 \quad , \quad W_3 = a^0 n_3 \quad , \quad W_4 = \rho^0 (a^0)^2 \quad , \quad W_5 = \rho^0 \qquad 8$$

but we shall not discuss the expression of k, which we know how to form and which is effectively used for predicting the boom in a homogeneous atmosphere, but which is fairly complex.

The discontinuities are not the only singularieties carried by the waves, since there are many others, one of the most interesting being the Dirac mass. Let us consider a family u_1^ε, u_2^ε, ..., u_n^ε of solutions of (H), dependent on parameter ε, and suppose that we have, whatever positive α may be,

$$\lim_{\varepsilon \to 0} u_i^\varepsilon = \hat{u}_i \quad , \quad |F| > \alpha \quad ,$$
$$\lim_{\varepsilon \to 0} \int_{-\alpha}^{\alpha} u_i^\varepsilon \, dF = \tilde{u}_i + \int_{-\alpha}^{\alpha} \hat{u}_i \, dF , \qquad 9$$

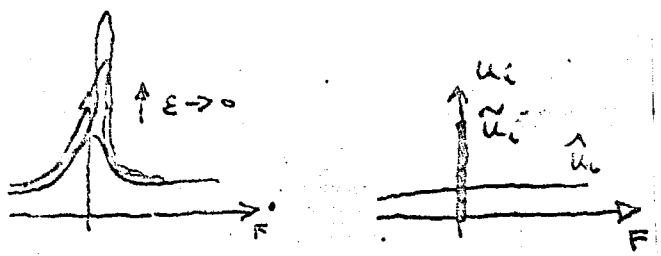


Fig. 8

so that, when ε tends to zero, u_i^ε tends to a function in the usual sense, denoted by $\hat{u}_i$, and augmented by a Dirac mass of intensity $\tilde{u}_i$, spread on F = 0. If u_i^ε is the solution of (H) for each ε, it can be shown that one should have

$$\tilde{u}_i = g w_i , \qquad 10$$

the w_i values being the same as in formula (5) and g still verifying the evolution equation (6). Without examining the demonstration of this result one is convinced of its plausibility by returning to formula (2) and noting that /5

$$u^\varepsilon = \frac{f^\varepsilon(r - at, R)}{r} , \qquad 11$$

is a sequence of the type considered if we have

$$f^\varepsilon(r - at, R) \to \tilde{g}(R) \, \delta(r - at) + \hat{f}(r - at, R) , \qquad 12$$

and hence

$$u^\varepsilon \to \tilde{u} \, \delta(r - at) + \hat{u} ,$$
$$\tilde{u} = \frac{\tilde{g}(R)}{r} \quad , \quad \hat{u} = \frac{\hat{f}(r - at, R)}{r} , \qquad 13$$

which is indeed the result announced in this particular case. This is highly significant for the shortwave theory and consequently for the supersonic boom theory; if we write

$$\lim_{\varepsilon \to 0} u_i^\varepsilon = \tilde{u}_i \, \delta(F) + \hat{u}_i , \qquad 14$$

we see that u_i^ε, and approximate representation of $\tilde{u}_i \delta(F) + \hat{u}_i$, has a shortwave structure, since, as soon as ε is small enough

the values of u_i^{ε} concentrate in an increasingly narrower band around F = 0, as ε becomes smaller. Without the necessity of going into details, it becomes clear that the main facts of the shortwave theory and in particular the supersonic boom theory become very striking. Let us conclude by stating without demonstration that the following generalized form of relation (2) imposes itself whenever it is necessary to describe a wave train of small width spread around a wave surface of a hyperbolic system:

$$u_i \sim g(\tau, R)\, f(F, R)\, W_i(\tau, R). \qquad 15$$

The functions of structure w_i are the same as in formula (5) describing the discontinuities; the signal shape as defined by f(F,R) is arbitrary, and finally the amplitude function $g(\tau,R)$ verifies equation (6). In the application of this formula to the prediction of the supersonic boom, the signal shape is determined by the Whitham function, related to the supersonic law of areas, as is well-known; the structure functions are extremely simple and analogous to (8); finally, the determination of the amplitude coefficient requires a complex numerical integration along the sound ray. Naturally, we are dealing only with the first stage, since it is still necessary to take into account the nonlinear effects of the dissipation.

So much therefore for the phenomenon of linear accumulation; we shall now come to the second component of the overall phenomenon, i.e., the nonlinear steepening of the signals. This elementary phenomenon is well-known in the theory of dimensionless time-dependent gas flows, and it is particularly apparent from the equation.

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0 \qquad 16$$

which shows that no linearization can be legitimately applied to such a phenomenon. The construction of a solution of (16) is classical, and it is known that this solution can be obtained graphically according to the method of Fig. 9., whose translation in formulas is expressed by

$$u = U(\omega), \qquad x = \omega + t\, U(\omega). \qquad 17$$

/6

Fig. 9 clearly shows that if a given signal is considered, its slope

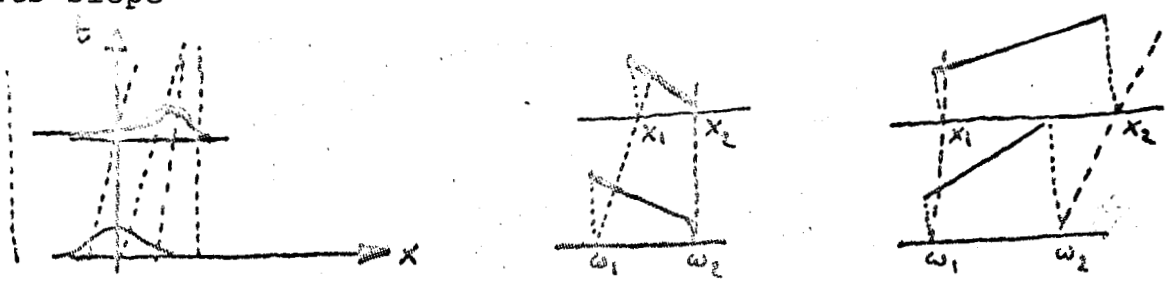


Fig. 9

has a tendency to become steeper where it is negative, whereas it tends to soften where it is positive. Even if one departs from a very gradual profile, the steepening phenomenon ends up developing overlaps at the end of a sufficient time period, and solution (16) ceases to be physically interpretable, since it leads one to assign three distinct values of u at certain points and at certain instants. The resolution of this difficulty is classical and is accomplished by inserting shock waves into the signal in place of the overlaps. Thus, if the dashed signal of Fig. 10 is the one predicted by the formula (17), it is the solid line signal with a shock wave that is physically interpretable. Obviously, the discontinuity could not be placed arbitrarily, and its emplacement is governed by the laws of shock waves well-known in gas dynamics. These laws are always stated in two parts and consist on the one hand of equality relations (for example, the Rankine-Hugoniot equations) and on the other hand of inequality relations (for example, the increase of entropy). It is instructive to dwell on this question of laws of shock for a little while, even though it is well-known by engineers, since what is perhaps a little less known is the manner in which it becomes integrated in the general context of the shortwave theory. Let us therefore digress from our subject, but only on the surface, and examine equation (16) as such; we can put it in the form of a conservation equation

Fig. 10

$$\frac{\partial U_0}{\partial t} + \frac{\partial U_1}{\partial x} = 0 \quad , \qquad U_0 = u \quad , \qquad U_1 = \frac{u^2}{2} \; ; \tag{18}$$

let us then designate by V a continuously differentiable function, zero on the contour of the region Ω ; if u is continuously differentiable, we can write

$$-\iint_{\Omega} \left(\frac{\partial U_0}{\partial t} + \frac{\partial U_1}{\partial x}\right) V \, dt \, dx = \iint_{\Omega} \left(U_0 \frac{\partial V}{\partial t} + U_1 \frac{\partial V}{\partial x}\right) dt \, dx . \tag{19}$$

One can then say that the "vector" $U = (U_0, U_1)$ is a weak solution of the conservation equation (18) if the following relation is verified for any function V continuously differentiable in Ω , zero on the edge of Ω (in order to avoid having a spurious contour integral) /7

$$\iint_{\Omega} \left(U_0 \frac{\partial V}{\partial t} + U_1 \frac{\partial V}{\partial x}\right) dt \, dx = 0 . \tag{20}$$

This method, invented in the 1930's by the Russian mathematician Sobolev and whose deep mathematical structure was studied in the 1940's under the name of theory of distributions by the French mathematician L. Schwartz, has now come into widespread use in the modern theory of partial differential equations. From our point of view, it is thus possible to give a meaning to equation (16) when u can be discontinuous, for example on a line Σ. Let us assume that Σ divides Ω into two parts, $\Omega+$ and $\Omega-$; we write

$$\iint_{\Omega^+} \left(U_0 \frac{\partial V}{\partial t} + U_1 \frac{\partial V}{\partial x}\right) dt\,dx + \iint_{\Omega^-} \left(U_0 \frac{\partial V}{\partial t} + U_1 \frac{\partial V}{\partial x}\right) dt\,dx = 0, \qquad 21$$

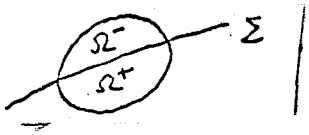


Fig. 11

and integrate by parts in $\Omega+$ and $\Omega-$ separately; if Σ has the slope

$$\frac{dx}{dt} = W_c,$$

we thus find

$$-\iint_{\Omega^+} \left(\frac{\partial U_0}{\partial t} + \frac{\partial U_1}{\partial x}\right) V\,dt\,dx - \iint_{\Omega^-} \left(\frac{\partial U_0}{\partial t} + \frac{\partial U_1}{\partial x}\right) V\,dt\,dx + \int_{\Sigma \cap \Omega} \left([U_1] - W_c [U_0]\right) dt = 0. \qquad 22$$

In this formula $[U_i]$ designates the discontinuity of U_i on Σ (positive x side minus negative x side). By first making V zero in $\Omega-$ (also on Σ), we see that (18) and hence (16) applies in $\Omega+$, then by making V zero n $\Omega+$ (and also on Σ) we see that (18) and hence (16^o) applies in $\Omega-$, and we are left with

$$\int_{\Sigma \cap \Omega} \left([U_1] - W_c [U_0]\right) dt = 0, \qquad 23$$

i.e., since V is arbitrary,

$$[U_1] - W_c [U_0] = 0. \qquad 24$$

Thus, according to (19), the notion of weak solution implies two things:

(I) (18) applies on either side of Σ,
(II) (24) applies on Σ. 25

This type of reasoning is completely general: let us consider a conservation equation (for example, one of Euler's equations)

$$\frac{\partial U_0}{\partial t} + \frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial x_2} + \frac{\partial U_3}{\partial x_3} = 0 \qquad 26$$

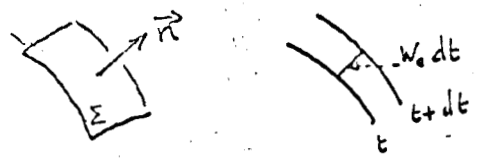


Fig. 12

and suppose that U_i are discontinuous on the surface Σ ; let us designate by $\vec{n}$ (n_1, n_2, n_3) the unit vector of the normal to Σ (directed from a certain side) and by w_c the normal velocity of displacement of Σ ; to say that $(U_0, U_1, U_2, U_3,)$ is a weak solution of equation (26) (according to a definition which generalizes (19)) implies two things:

/8

(I) (26) applies on either side of Σ ,
(II) relation (28) below applies on Σ , 27

$$n_1[U_1] + n_2[U_2] + n_3[U_3] - w_c[U_0] = 0 . \qquad 28$$

In this formula, $[U_i]$ designates the discontinuity of U_i (side $\vec{n}$ minus side $-\vec{n}$). The Rankine-Hugoniot relations of gas dynamics can best be formed by considering that U_0, U_1, U_2, U_3 are vectors with five components

$$U_0 = \left\{ \rho,\ \rho u_1,\ \rho u_2,\ \rho u_3,\ \rho\left(e + \frac{u_1^2+u_2^2+u_3^2}{2}\right) \right\},$$

$$U_i = \left\{ \rho u_i,\ \rho u_1 u_i + p\,\delta_{1i},\ \rho u_2 u_i + p\,\delta_{2i},\ \rho u_3 u_i + p\,\delta_{3i},\ \left[\rho\left(e + \frac{u_1^2+u_2^2+u_3^2}{2}\right) + p\right] u_i \right\} \qquad 29$$

with, for example, $\delta_{11} = 1$, $\delta_{12} = \delta_{13} = 0$. If this method is applied to equation (16) in its conservation form (18), we find the corresponding condition of shock

$$\left[\frac{u^2}{2}\right] - w_c[u] = 0 , \qquad 30$$

i.e.,

$$u^+ + u^- = 2w_c , \qquad 31$$

which can be interpreted in terms of velocities, if u represents a velocity, by saying that the velocity of displacement of the shock is an arithmetic mean of the upstream and downstream velocities. This we recognize to be a classical statement in gas dynamics and one pertaining to shocks of weak intensity. It is thus possible to characterize the weak solutions of (16) by specifying that (31) takes place on each discontinuity line. Figure 13

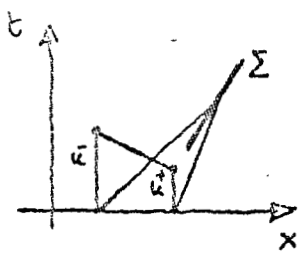
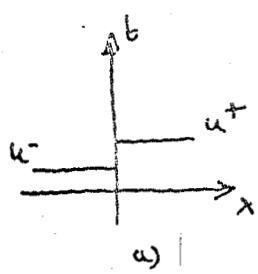


Fig. 13



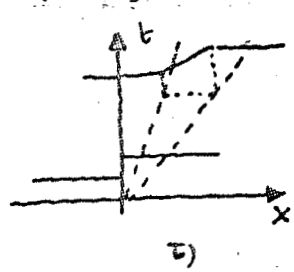


Fig. 14

specifies the geometric interpretation of the relation of shock. Yet it is easy to see that the simple equality relations are insufficient by examining the example of Fig. 14, which shows 2 weak solutions corresponding to the same profile u(x) at instant t=0 for equation (16). The first solution, unacceptable physically, would correspond to a decompression shock in gas dynamics, whereas the second would correspond to a Prandtl-Meyer expansion, and this is the one which should be retained. The spurious solutions are eliminated by completing the equality relations of the shock wave with one or several inequality relations. Thus, it is known in gas dynamics that (26) with the choice (29) implies the complementary conservation relation

$$\frac{\partial \phi_0}{\partial t} + \frac{\partial \phi_1}{\partial x_1} + \frac{\partial \phi_2}{\partial x_2} + \frac{\partial \phi_3}{\partial x_3} = 0$$

$$\phi_0 = \rho s \,, \quad \phi_1 = \rho s u_1 \,, \quad \phi_2 = \rho s u_2 \,, \quad \phi_3 = \rho s u_3 \qquad (32)$$

where s designates the specific entropy. It happens that the shock relation formally associated with this sixth conservation equation is incompatible with those associated with the first five. Indeed, this sixth conservation equation is known to give rise to a shock relation in the form of an inequality which expresses the fact that the entropy does not decrease upon passage of the shock wave, i.e., /9

$$N_1[\phi_1] + N_2[\phi_2] + N_3[\phi_3] - W_a[\phi_0] \geq 0 \,. \qquad (33)$$

The name of solutions with shock is given to weak solutions which verify one or several complementary inequalities of this type. The appropriate choice of these complementary relations raises a difficult mathematical problem even if, under more usual circumstances, this choice is physically evident (let us note that the physical evidence is already less clear in magnetogasdynamics). One method which is valid for Euler equations (assuming that the answer is not known otherwise) consists in considering the solutions with shock as limits of solutions of Navier-Stokes equations when the viscosity and conductivity are evanescent. Let us therefore return to equation (16); it seems reasonable to consider that its solutions with shock are, when ν tends to zero, limits of solutions of the Burgers equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \,. \qquad (34)$$

any solution of this equation is also a solution of the following one:

$$\frac{\partial}{\partial t}\left(\frac{u^2}{2}\right) + \frac{\partial}{\partial x}\left\{\frac{u^3}{3} - \nu u \frac{\partial u}{\partial x}\right\} + \nu\left(\frac{\partial u}{\partial x}\right)^2 = 0 \,, \qquad (34)$$

so that if v is a continuously differentiable function becoming zero on the edge of Ω, we have

$$\iint_{\Omega}\left(\frac{u^2}{2}\frac{\partial v}{\partial t}+\left(\frac{u^3}{3}-\nu u\frac{\partial u}{\partial x}\right)\frac{\partial v}{\partial x}\right)dt\,dx=\iint_{\Omega}\nu\left(\frac{\partial u}{\partial x}\right)^2 v\,dt\,dx. \qquad 35$$

Let u^ν be a family of solutions of (34) tending to a solution with shock of (16) when ν tends to zero; it can be shown that

$$\lim_{\nu\to 0}\ \nu\iint_{\Omega} u^\nu\frac{\partial u^\nu}{\partial x}\frac{\partial v}{\partial x}\,dt\,dx=0 \qquad 36$$

so that one is led to define a solution with shock of (16) as a function of u verifying the following equality and inequality relations:

$$\iint_{\Omega}\left(u\frac{\partial v}{\partial t}+\frac{u^2}{2}\frac{\partial v}{\partial x}\right)dt\,dx=0 \qquad 37$$

$$\iint_{\Omega}\left(\frac{u^2}{2}\frac{\partial v}{\partial t}+\frac{u^3}{3}\frac{\partial v}{\partial x}\right)dt\,dx\leq 0 \qquad (v\geqslant 0) \qquad 38$$

for any continuously differentiable function v zero at the edge of region Ω and arbitrary. The reasoning already developed implies that if u verifies these two relations and undergoes a discontinuity on Σ, this implies the following three conclusions:

$$\text{(I)}\quad \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0 \quad , \quad \text{on either side of } \Sigma, \qquad /10$$

$$\text{(II)}\quad \left[\frac{u^2}{2}\right]-W_c[u]=0 \quad , \quad \text{on } \Sigma, \qquad 39$$

$$\text{(III)}\quad \left[\frac{u^3}{3}\right]-W_c\left[\frac{u^2}{2}\right]\leq 0 \quad , \quad \text{on } \Sigma.$$

It is easy to see that (II) and (III) imply

$$\left[(u-W_c)^3\right]\leq 0 \ , \qquad 40$$

and thus we have, for all the conditions of shock associated with (16),

$$\text{(I)}\quad u^+ + u^- = 2W_c \ , \qquad 41$$

$$\text{(II)}\quad u^+ \leq W_c \leq u^- \ .$$

In order to complete this analysis of equation (16), let us recall the following result, well-known by engineers who make calculations of the supersonic boom: the site of a shock wave

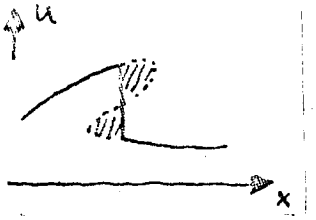


Fig. 15

in a profile with overlap is such that the two shaded areas of Fig. 15 are equal. Here is a very simple demonstration of this result, based on the observation that it is sufficient to demonstrate the persistence of the property in question in the course of time. Let us consider configuration a of Fig. 16, for which the property is true, and let us take the curve C_{to} in the shape of a figure eight, as the initial given quantity for a solution in the classical sense of (16), i.e.

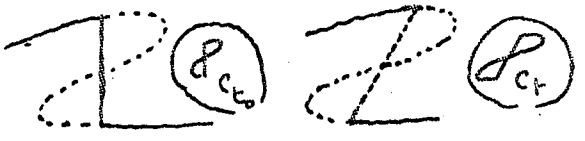


Fig. 16

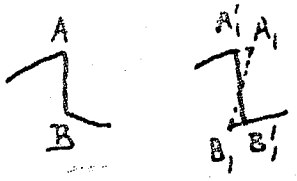


Fig. 17

a solution described by formulas (17); at a later instant, the profile has come to C_t in accordance with configuration b of Fig. 16, and we have

$$\oint_{C_t} u\,dx = \oint_{C_{to}} U(\omega)\,d\omega + (t-t_o)\int_{C_{to}} U(\omega)\,dU(\omega), \qquad 42$$

but the first integral is zero by hypothesis, whereas the second integral is none other than the variation of $U(\omega)$ along the figure eight constituted by C_{to}, and this variation is obviously zero. Finally, the two infinitely close configurations of Fig. 17 remain to be examined; in this sense, the shock relation expresses the fact that the common abscissa of A_1' and B_1' is the geometrical mean of the abscissas of A_1 and B_1, which implies the equality of the areas and completes the demonstration.

After this long digression, let us return to the theory of short waves in order to include the nonlinear effects therein. Let us consider a system of conservation equations.

/11

$$\frac{\partial U_0}{\partial t} + \frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial x_2} + \frac{\partial U_3}{\partial x_3} = 0\ , \quad \left(\frac{\partial U_\alpha}{\partial x_\alpha} = 0\ , x_0 = t\right), \qquad 43$$

where U_a are vectors having the same number of components, given for example by (29) if we are dealing with gas dynamics. Later, the Greek subscripts a will take the four values 0, 1, 2, 3, whereas the Latin subscripts i will take the three values 1, 2, 3. In general, U_i are functions of U_o, i.e.,

$$U_i = U_i(U), \qquad 44$$

U being written in place of U_o; we can verify that this is indeed the case with the choice (29); let us give ourselves a solution U_a^o of the system (43), for example the one which describes the undisturbed atmosphere, and consider a neighboring solution

$$U_\alpha = U_\alpha^\circ + U_\alpha^1 , \qquad 45$$

assuming that U_a^1 is small compared to U_a^o, in a certain sense. By differentiating (44), we obtain

$$dU_i = A_i\, dU \qquad 46$$

where A_i are matrices dependent on U; as a result of (45) we can write

$$U_i^1 = A_i^\circ\, U^1 \quad , \quad (A_i^\circ = A_i(U^\circ)) , \qquad 47$$

and we see that, through a formal linearization, the U's verify the system of linear partial differential equation

$$\frac{\partial U^1}{\partial t} + \sum_{i=1}^{3} A_i^\circ \frac{\partial U^1}{\partial x_i} + \sum_{i=1}^{3} \frac{\partial A_i^\circ}{\partial x_i}\, U^1 = 0 \ . \qquad 48$$

If we are dealing with gas dynamics, (48) is the system of equations of acoustics relative to the homogeneous atmosphere if A_1^o are constant, and relative to the inhomogeneous atmosphere if A_1^o are variable. It follows from this remark that the developments which will follow contain the whole of the supersonic boom theory as an application, excluding the dissipation effects. Let us now suppose that U^1 has a shortwave structure.

$$U^1 \simeq \tilde{U}^1(F, x_\alpha) , \qquad 49$$

with a strong dependence with respect to F and weak dependence with respect to X<; we write

$$\frac{\partial U^1}{\partial x_\alpha} = \boxed{\frac{\partial \tilde{U}^1}{\partial F}\frac{\partial F}{\partial x_\alpha}} + \frac{\partial \tilde{U}^1}{\partial x_\alpha} , \qquad 50$$

noting that the framed terms are predominant, and we substitute into (48) to obtain

$$\boxed{\left(\frac{\partial F}{\partial t} + \sum_{i=1}^{3} A_i^\circ \frac{\partial F}{\partial x_\alpha}\right)\frac{\partial \tilde{U}^1}{\partial F}} + \frac{\partial \tilde{U}^1}{\partial t} + \sum_{i=1}^{3} A_i^\circ \frac{\partial \tilde{U}^1}{\partial x_i} + \sum_{i=1}^{3} \frac{\partial A_i^\circ}{\partial x_i}\, \tilde{U}^1 = 0 \ . \qquad 51$$

Again, the framed terms are predominant and hence should cancel out, which implies

$$\left(\frac{\partial F}{\partial t} + \sum_{i=1}^{3} A_i^\circ \frac{\partial F}{\partial x_i}\right)\frac{\partial \tilde{U}^1}{\partial F} = 0 \ . \qquad 52$$

Let n ($\vec{n}_1=n_2$, n_3) be the unit vector with F=const and w_c the velocity of normal displacement of F=const; we have

/12

$$\frac{\partial F}{\partial t} = -\lambda w_c \quad , \quad \frac{\partial F}{\partial x_i} = \lambda n_i \qquad 53$$

and thus

$$\left(\sum_{i=1}^{3} n_i A_i^\circ - w_c 1\right) \frac{\partial \tilde{U}^1}{\partial F} = 0 \, , \qquad 54$$

designating the unit matrix by 1. This implies two things:

(i) $\det\left(\sum_i n_i A_i^\circ - w_c 1\right) = 0$,

(ii) $\frac{\partial \tilde{U}^1}{\partial F}$ is the eigenvector to the right of $\left(\sum_i n_i A_i^\circ - w_c 1\right)$. 55

Point (i) signifies that F=const is a wave surface for equation (48), in the sense of the theory of characteristics; this wave surface can support a discontinuity of U^1 provided that $[U^1]$ be the eigenvector to the right of the matrix $\sum_i k_i A_i^\circ - w_c 1$; point (ii) signifies therefore that the profile $\frac{\partial U_1}{\partial F}$ has the same structure as a discontinuity. Thus we are led to specify (49) as follows:

$$U^1 \sim f(F, x_\alpha) W(x_\alpha) \, , \qquad 56$$

with

$$\left(\sum_{i=1}^{3} n_i A_i - w_c 1\right) W = 0 \, . \qquad 57$$

This means that if a solution of the linear equation has a short-wave structure, it is spread in the vicinity of a characteristic surface or a wave surface, and that it is proportional to a discontinuity tolerated upon passage of this wave surface. For the time being, however, we still know nothing about the function of amplitude f (F, X_α). Let us therefore try to improve the preceding representation by setting

$$U^1 \sim f W + \tilde{U}^{1,2} ; \qquad 58$$

neglecting terms smaller than the terms retained, we obtain

$$\left(\frac{\partial F}{\partial t} 1 + \sum_i \frac{\partial F}{\partial x_i} A_i^\circ\right) \frac{\partial \tilde{U}^{1,2}}{\partial F} + \left(\frac{\partial f}{\partial t} 1 + \sum_i \frac{\partial f}{\partial x_i} A_i^\circ\right) \cdot W + \left(\frac{\partial W}{\partial t} + \sum_i A_i^\circ \cdot \frac{\partial W}{\partial x_i} + \sum_i \frac{\partial A_i^\circ}{\partial x_i} \cdot W\right) f = 0 . \qquad 59$$

In order for this equation to be solvable, it is necessary to satisfy a compatibility condition which is obtained by multiplying (by matrices) on the left by the eigenvector on the left T, i.e.,

$$T \cdot \left(\sum_i n_i A_i^\circ - w_c 1\right) = 0 \qquad 60$$

and we thus obtain

$$\frac{\partial f}{\partial \sigma} T.W + \sum_i \frac{\partial f}{\partial x_i} T.A_i^\circ.W + \left(T.\frac{\partial W}{\partial t} + \sum_i T\, A_i^\circ \frac{\partial W}{\partial x_i} + \sum_i T.\frac{\partial A_i^\circ}{\partial x_i}.W \right) f = 0. \quad 61$$

Since W and T are known vectors, the above is a scalar equation for the only unknown (F,$\varkappa$). Generally, it can be managed so that T and W verify the normalization condition

$$T.W = 1 \quad 62$$

and this is particularly the case in gas dynamics. It is shown that with the notations of (53),

$$T.A_i^\circ.W = \frac{dx_i}{dt}, \text{ on a ray} \quad 63$$

/13

and that one can therefore write

$$\frac{\partial}{\partial t} + T.A_i^\circ.W \frac{\partial}{\partial x_i} = \frac{\partial}{\partial \tau}, \quad 64$$

designating by $\frac{\partial}{\partial \tau}$ a time derivative by following the wave along a ray (sound ray in acoustics). Thus f verifies the ordinary differential equation

$$\frac{\partial f}{\partial \tau} + k f = 0 \quad , \quad k = T.\frac{\partial W}{\partial t} + \sum_{i=1}^{3} T.A_i^\circ.\frac{\partial W}{\partial x_i} + \sum_{i=1}^{3} T.\frac{\partial A_i^\circ}{\partial x_i}.W, \quad 65$$

and, since k is independent of F, (56) can be specified in the following form

$$U^1 \sim g(\tau, \alpha)\, f(F, \alpha)\, W(\tau, \alpha), \quad 66$$

where f this time is an arbitrary function of F on each ray, independent of τ, and where the amplitude function g, independent of F, still verifies the equation

$$\frac{\partial g}{\partial \tau} + k g = 0 . \quad 67$$

We thus find again all the elements of the linear shortwave theory posed in the beginning, but this can still be specified in the important case where the matrices A_i° are symmetrical. Indeed, in this case we have T = W, and consequently

$$k = \frac{1}{2} \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left(W.A_i^\circ.W \right) + \frac{1}{2} \sum_{i=1}^{3} W.\frac{\partial A_i^\circ}{\partial x_i}.W. \quad 68$$

Furthermore, if on the left equation (48) is multiplied (by matrices) by U, we obtain an energetic identity

$$\frac{\partial}{\partial t}\left(\frac{1}{2} U^1.U^1 \right) + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(\frac{1}{2} U^1.A_i^\circ.U^1 \right) + \sum_{i=1}^{3} U^1.\frac{\partial A_i^\circ}{\partial x_i}.U^1 = 0 \quad 69$$

which presents structural analogies with the expression of k. We can go further by integrating this equation, with τ and R fixed at F, over an interval which contains the entire significant part of the wave train; we have

$$\frac{\partial}{\partial t}\int_{\alpha}^{\beta}\frac{1}{2}U^1\cdot U^1\,dF+\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\int_{\alpha}^{\beta}\frac{1}{2}U^1\cdot A_i^0\cdot U^1\,dF+\int_{\alpha}^{\beta}\frac{1}{2}\sum_{i=1}^{3}U^1\cdot\frac{\partial A_i^0}{\partial x_i}\cdot U^1\,dF=0, \qquad 70$$

and, substituting (66) into this relation, we again find the differential equation (67). This explains why the evolution equation is sometimes established by utilizing energetic arguments whose justification is somewhat devious, since, in the case of gas dynamics and with the natural choice (29), the matrices A_i^0 are not symmetrical. Nevertheless, there is a profound reason for the success of these energetic methods: a system of conservation equations can be formed which is equivalent to the natural system of gas dynamics, for which the matrices A_i are symmetrical; the equations of gas dynamics are said to be symmetrizable. The choice to be made is as follows: the five-component vector

$$V=\left\{\frac{h}{T}-s-\frac{u_1^2+u_2^2+u_3^2}{2T},\ \frac{u_1}{T},\ \frac{u_2}{T},\ \frac{u_3}{T},\ -\frac{1}{T}\right\}=\{V_0,\ V_1,\ V_2,\ V_3,\ V_4\} \qquad 71$$

is considered by designating the specific enthalpy by h and the temperature by T; if p is the pressure, the expression $\phi=\frac{p}{T}$ is considered to be a function of the variables $v_0,..v_4$, and the same is done with the expression $\phi_i=\frac{pu_i}{T}$ it is then necessary to make the following choice /14

$$U_0=\left\{\frac{\partial\phi}{\partial V_0},\frac{\partial\phi}{\partial V_1},\frac{\partial\phi}{\partial V_2},\frac{\partial\phi}{\partial V_3},\frac{\partial\phi}{\partial V_4}\right\},$$
$$U_i=\left\{\frac{\partial\phi_i}{\partial V_0},\frac{\partial\phi_i}{\partial V_1},\frac{\partial\phi_i}{\partial V_2},\frac{\partial\phi_i}{\partial V_3},\frac{\partial\phi_i}{\partial V_4}\right\},\qquad i=1,2,3. \qquad 72$$

It suffices that this simple choice be possible in order that, even with the natural choice (29), we have

$$\sum_{i=1}^{3}T\cdot A_i^0\cdot\frac{\partial W}{\partial x_i}=\frac{1}{2}\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(TA_i^0W\right)-\frac{1}{2}\sum_{i=1}^{3}T\cdot\frac{\partial A_i^0}{\partial x_i}\cdot W, \qquad 73$$

which makes it possible to write coefficient k in the evolution equation as

$$k=\frac{1}{2}\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(T\cdot A_i^0\cdot W\right)+\frac{1}{2}\sum_{i=1}^{3}T\cdot\frac{\partial A_i^0}{\partial x_i}\cdot W. \qquad 74$$

Let us now consider the problem of nonlinear low amplitude shortwaves; we shall retain (43), (44), (45), (46), and (49), but shall examine more closely the linearization process leading to

(48). First of all, the Taylor expansion (47) can be carried further to give

$$U_i(U^0+U^1) = A_i^0 U^1 + B_i^0(U^1,U^1) + \cdots , \qquad 75$$

designating by $B_i^0 U^1$, U^1 a vector, quadratic form of U^1, whose coefficients depend on U^0. We shall also use the following notation for the polar form

$$2B_i^0(U^1,V^1) = A_i^1(U^0,U^1)\,V^1 \qquad 76$$

by designating by A_i^1 a matrix analogous to A_i^0 which depends on U^0 and linearly on U. We substitute (75) into the conservation system (43) and get

$$\frac{\partial U^1}{\partial t} + \sum_{i=1}^{3}\left\{A_i^0\frac{\partial U^1}{\partial x_i} + 2B_i^0\left(U^1,\frac{\partial U^1}{\partial x_i}\right)\right\} + \sum_{i=1}^{3}\left\{\frac{\partial A_i^0}{\partial x_i}U^1 + \frac{\partial B_i^0}{\partial x_i}(U^1,U^1)\right\} = 0, \qquad 77$$

neglecting the terms which are formally third order. We designated by $\frac{\partial B_i^0}{\partial x_i}(U^1,U^1)$ a quadratic form which is obtained from $B_i^0(U^1,U^2)$ by differentiating the coefficients, so that we have

$$\left|\frac{\partial B_i^0}{\partial x_i}(U^1,U^1)\right| \ll \left|\frac{\partial A_i^0}{\partial x_i}U^1\right|, \qquad 78$$

and so that we can confine ourselves to the consideration of the equation

$$\frac{\partial U^1}{\partial t} + \sum_{i=1}^{3}\left(A_i^0 + A_i^1\right)\frac{\partial U^1}{\partial x_i} + \sum_{i=1}^{3}\frac{\partial A_i^0}{\partial x_i}U^1 = 0, \qquad 79$$

in which we substitute (49) and (50) to obtain

$$\left\{\frac{\partial F}{\partial t}1 + \sum_{i=1}^{3}\frac{\partial F}{\partial x_i}\left(A_i^0+A_i^1\right)\right\}\cdot\frac{\partial\tilde{U}^1}{\partial F} + \frac{\partial\tilde{U}^1}{\partial t} + \sum_{i=1}^{3}\left(A_i^0+A_i^1\right)\frac{\partial\tilde{U}^1}{\partial x_i} + \sum_{i=1}^{3}\frac{\partial A_i^0}{\partial x_i}U^1 = 0. \qquad 80$$

as above in (51), the framed terms are predominant, and we reach the following double conclusion

/15

(I) and $\left\{\sum_{i=1}^{3}n_i\left(A_i^0+A_i^1\right) - W_c 1\right\} = 0$,

(II) $\frac{\partial\tilde{U}^1}{\partial F}$ is the eigenvector to the right of $\sum_{i=1}^{3}n_i\left(A_i^0+A_i^1\right) - W_c 1$. 81

Point (I) means that F = const is a wave surface for the system

$$\frac{\partial V}{\partial t} + \sum_{i=1}^{3}\left(A_i^0 + A_i^1\right)\frac{\partial V}{\partial x_i} = 0, \qquad 82$$

and hence that the propagation of this wave is affected by the perturbations. The reasoning already presented can be continued so as to obtain the representation

$$U^1 \sim g(\tau, R)\, f(F, R)\, W(\tau, R), \qquad 83$$

R designating this time a ray of propagation associated with the equation

$$\det\left\{\sum_i \frac{\partial F}{\partial x_i}\left(A_i^0 + A_i^1\right) + \frac{\partial F}{\partial t} 1\right\} = 0, \qquad 84$$

T and W being the eigenvectors on the right and left, in the following sense

$$\left\{\sum_{i=1}^{3} n_i \left(A_i^0 + A_i^1\right) - w_c 1\right\} \cdot W = 0,$$
$$T \cdot \left\{\sum_{i=1}^{3} n_i \left(A_i^0 + A_i^1\right) - w_c 1\right\} = 0, \qquad 85$$

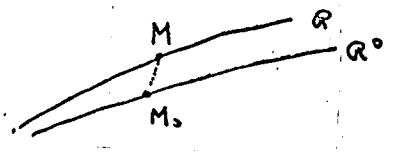
with normalization (62). As above, the wave form of the signal, defined by function f, is arbitrary on each ray, and amplitude function g verifies an evolution equation

$$\frac{\partial g}{\partial \tau} + k\, g = 0, \qquad 86$$

$$k = T \cdot \frac{\partial W}{\partial t} + \sum_{i=1}^{3} T \cdot \left(A_i^0 + A_i^1\right) \cdot \frac{\partial W}{\partial x_i} + \sum_{i=1}^{3} T \cdot \frac{\partial A_i^0}{\partial x_i} \cdot W, \qquad 87$$

$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} + \sum_{i=1}^{3} T \cdot \left(A_i^0 + A_i^1\right) \cdot W \frac{\partial}{\partial x_i}. \qquad 88$$

The representation obtained is uselessly complex, in the sense that the low amplitude of the perturbations was not utilized as much as it could have been. Let us note that rays R can be placed in biunivocal correspondence with corresponding rays R^o of the acoustic approximation, and in a manner so that at the same instant and for the same value of F, the corresponding points M and M^o are neighboring ones. One can thus advantageously substitute the representation F, τ, R by a representation F, t^o, R^o whose correspondence with the Cartesian coordinates is known, independently of the desired solution U^1. Noting that product fg verifies equation (86) in the same way as g one is thus led to set



$$g(\tau, R)\, f(F, R) = f^o(F, \tau^o, R^o), \qquad 89$$

and try to determine what becomes of the evolution equation. Let T^o and W^o be the values taken by T and W in the acoustical approximation in τ^o, R^o, assuming that the dependence relative to F is

negligible on the quantities pertaining to the unperturbed atmosphere. Let us note that /16

$$\frac{\partial}{\partial \tau^\circ} = \frac{\partial}{\partial t} + \sum_{i=1}^{3} T^\circ A_i^\circ W^\circ \frac{\partial}{\partial x_i}, \qquad 90$$

so that the evolution equation can be written as

$$\frac{\partial f^\circ}{\partial \tau^\circ} + \boxed{\sum_{i=1}^{3} \lambda_i(U^1) \frac{\partial f^\circ}{\partial x_i}} + k^\circ f^\circ = 0 \qquad 91$$

$$k^\circ = \eta^\circ \frac{\partial W^\circ}{\partial t} + \sum_{i=1}^{3} T^\circ A_i^\circ \frac{\partial W^\circ}{\partial x_i} + \sum_{i=1}^{3} T^\circ \frac{\partial A_i^\circ}{\partial x_i} W^\circ, \qquad 92$$

the terms of greater order in k being neglected. To evaluate this framed term, let us again take the representation

$$f^\circ = \tilde{f}^\circ(F^\circ; x^a) = \tilde{f}(F, x^a) \qquad 93$$

we have

$$\sum_{i=1}^{3} \lambda_i(U^1) \frac{\partial f^\circ}{\partial x_i} = \boxed{\left(\sum_{i=1}^{3} \lambda_i(U^1) \frac{\partial F^\circ}{\partial x_i}\right) \frac{\partial \tilde{f}^\circ}{\partial F^\circ}} + \sum_{i=1}^{3} \lambda_i(U^1) \frac{\partial \tilde{f}^\circ}{\partial x_i}, \qquad 94$$

and let us note that the framed term is predominant, so that the evolution equation reduced to significant terms is written as

$$\frac{\partial f^\circ}{\partial \tau^\circ} + \left(\sum_{i=1}^{3} \lambda_i(U^1) \frac{\partial F^\circ}{\partial x_i}\right) \frac{\partial \tilde{f}^\circ}{\partial F^\circ} + k^\circ f^\circ = 0 . \qquad 95$$

All we now have to evaluate is the coefficient of $\frac{\partial f^\circ}{\partial F^\circ}$, but this is remarkably simple, since we have

$$\sum_{i=1}^{3} \lambda_i(U^1) \frac{\partial F^\circ}{\partial x_i} \cong \sum_{i=1}^{3} T^\circ \left(\frac{\partial F}{\partial x_i}(A_i^\circ + A_i^1)\right) \cdot W - \sum_{i=1}^{3} T^\circ A_i^\circ W^\circ = -\frac{\partial F}{\partial t} + \frac{\partial F^\circ}{\partial t} . \qquad 96$$

Finally, if it is arranged that we have, in the range of useful variation of F,

$$\frac{\partial F}{\partial x_i} = h_i \qquad \frac{\partial F^\circ}{\partial x_i} = h_i^\circ \qquad 97$$

it appears that the evolution equation, taking the nonlinear effects into account, assumes a remarkably simple form, i.e.,

$$\frac{\partial f^\circ}{\partial \tau^\circ} + (w_c - w_c^\circ) \frac{\partial f^\circ}{\partial F^\circ} + k^\circ f^\circ = 0 \qquad 98$$

where $w_c - w_c^\circ$ designates the variation of the propagation velocity (of the wave) due to the perturbation U^1.

This result, which has a very general significance, covers in applications all that is involved in the supersonic boom theory except focalization; the supersonic boom of focalization, otherwise called the superboom, definitely eludes the shortwave theory. Therefore, because of the theoretical and practical importance of the preceding result, the latter should be considered for a few moments. Let us first interpret it in terms of the pictorial language. The first important point is that a wave train of small width (in particular, the supersonic boom) may be followed on each ray of the acoustic approximation, the rays being completely uncoupled from one another from this point of view. This proposition was utilized for a long time by engineers before its formal demonstration was given. It was therefore imperative to adduce its formal proof and to convince oneself of /17
its great degree of generality. The second point is that the perturbation in the wave train is described by formulas of the type

$$U^1 \simeq f(\tau, \xi; R)\, W(\tau, R), \qquad 99$$

where $W(\tau, R)$ is a vectorial function which can be determined by analyzing the structure of the discontinuities for the discontinuous solutions of the acoustic (linear) approximation; this function of structure is attached to the central (acoustic) wave of the train. Variable ξ represents the distance, in the wave train, from a point of the latter to the central wave in relation to a ray. Fig. 18 gives a pictorial explanation of this construction: thus, if the central wave passes at P on ray R at instant τ, point Q is at the same instant at a distance $\xi = PQ$ on the normal to the wave at P. The last point to be mentioned is that the function of amplitude f is governed, on each ray R, by an evolution equation

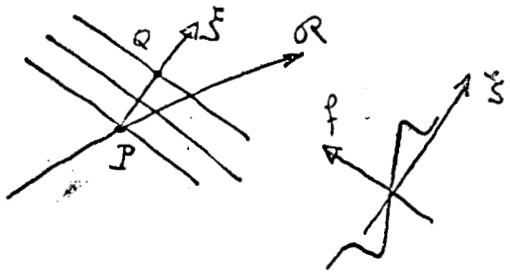


Fig. 18

$$\frac{\partial f}{\partial \tau} + \Gamma f \frac{\partial f}{\partial \xi} + k f = 0, \qquad 100$$

which has the structure of a Burgers equation without a dissipation term. The coefficient Γ, attached to the wave on each ray, has a meaning which can be explained by examining Fig. 19. Let us suppose that the central wave of the train for the acoustic approximation is at P^o on ray R^o at instant τ, and that for the perturbed medium it is at P on ray R at the same instant; let us also suppose that the wave located at distance ξ from the central wave is, for the acoustic approximation, at Q^o on the normal to the wave at P^o and for the perturbed medium at Q on the normal

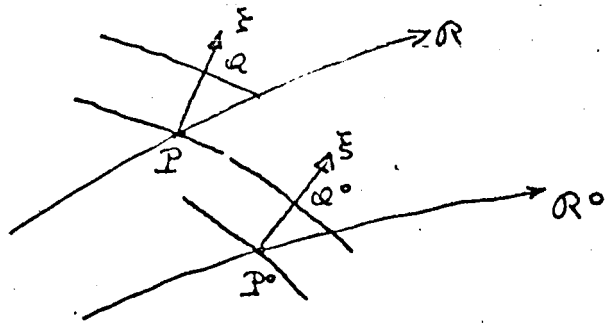


Fig. 19

to the wave at P, at the same instant τ ; and consider that we have (w_c being the normal propagation velocity of the wave)

$$W_c(Q,\tau) - W_c^\circ(Q^\circ,\tau) = \int f \qquad 101$$

and, in the application of this formula, there is no need to consider the difference in orientation between the acoustic wave and the perturbed wave (this latter point results from estimates which we have not presented). In gas dynamics, if f designates the perturbation of the component (normal to the wave) of the velocity vector, we have $\Gamma = \frac{\gamma+1}{2}$, denoting by γ the ratio of the specific heats of the (perfect) gas. This statement can therefore be interpreted by saying the following: the nonlinearity effect is reduced to a phenomenon of nonlinear convection along the rays of the acoustic approximation. This is in fact a differential convection relative to the propagation of the acoustic wave, and the convection velocity is equal to the increase of /18
the velocity of (normal) propagation due to the perturbations; any effect of lateral convection (which is actually present) may be neglected as a first approximation. It will be noted that this nonlinear convection phenomenon produces a distortion of the signal, which steepens and may develop shock waves in some of its parts, while it spreads in other parts. Onto this distortion effect, which is actually accompanied by dissipation in the shock waves and hence by attenuation, is superimposed a phenomenon of evolution of amplitude without distortion, characterized in the evolution equation by the factor k, and which is due to the variation in the volume of the cells in which the energy is trapped in the acoustic approximation. In the literature on the supersonic boom, one finds certain discrepancies concerning the evaluation of this second effect, i.e., concerning the value of coefficient k (in the case of an inhomogeneous atmosphere with wind, the case of an atmosphere without wind being unambiguous). These discrepancies are due to the use of physical intiuition in an area where the physical phenomena are complex; this underlines the importance of the role of mathematics and shows the advantage of approaching problems posed by technology from a standpoint which may at first appear very abstract in comparison with technical requirements.

After this excessively long discussion of the first two elementary phenomena which dominate the theory of the supersonic boom, a few brief remarks will suffice for a discussion of the

dissipation effects, since we already dealt with them in connection with the Burgers equation. In fact, it will suffice to prove that in the case of the supersonic boom, the set of the three phenomena is governed by a Burgers equation. Our point of departure consists in the obvious remark that the dissipation effects will be included if the theory of short waves is reformulated by starting from Navier-Stokes equation instead of Euler equations. It can be readily shown that with the choice (71) and by setting

$$(\phi_0, \phi_1, \phi_2, \phi_3) = \left(\frac{1}{T}, \frac{\rho u_1}{T}, \frac{\rho u_2}{T}, \frac{\rho u_3}{T}\right), \qquad 102$$

the Navier-Stokes equations are put in the form

$$\frac{\partial}{\partial x_\alpha}\left(\phi_\alpha(V)\right) = \frac{\partial}{\partial x_K}\left(L_{KJ}\frac{\partial V}{\partial x_J}\right), \qquad 103$$

with summation in the first term on subscript α ranging from 0 to 3 and summation in the second member on the set of two Latin indices ranging from 1 to 3. For each subscript couple, L_{kt} designates a matrix 5x5 operating on the vector space of the variable v. Let us suppose that v is the sum of an unperturbed component which is the solution of the equations and of a perturbation having the shortwave structure, i.e.,

$$V = V^\circ + \tilde{V}^1(F, x_\alpha) \qquad 104$$

By substituting in the second member of (103) and keeping only the significant perturbing terms, we obtain the expression

/19

$$\frac{\partial}{\partial x_K}\left(L_{KJ}\frac{\partial V}{\partial x_J}\right)\Big\}^1 = \frac{\partial F}{\partial x_K} L^\circ_{KJ}\frac{\partial F}{\partial x_J}\frac{\partial^2 \tilde{V}^1}{\partial F^2}, \qquad 105$$

with summation over the set of subscripts K and J, ranging from 1 to 3. Let us now consider the first member of (103), and for this purpose, set

$$\frac{\partial^2 \phi_\alpha}{\partial v \partial v} = A_\alpha \; ; \qquad 106$$

we obtain a Taylor expansion analogous to (75)

$$\phi_\alpha - \phi^\circ_\alpha = A^\circ_\alpha V^1 + 2B_\alpha(V^1, V^1) + \cdots \qquad 107$$

which makes it possible to write the perturbation of the first member of (103) in the form

$$\left\{\frac{\partial}{\partial x_\alpha}\left(\frac{\partial \phi_\alpha}{\partial v}\right)\right\}^1 = A^\circ_\alpha \frac{\partial V^2}{\partial x_\alpha} + 2B^\circ_\alpha\left(V^1, \frac{\partial V^2}{\partial x_\alpha}\right) + \frac{\partial A^\circ_\alpha}{\partial x_\alpha} V^1 + \frac{\partial B^\circ_\alpha}{\partial x_\alpha}(V^1, V^1). \qquad 108$$

We thus obtain, for the equation with perturbations,

$$\frac{\partial F}{\partial x_\alpha} A^\circ_\alpha \frac{\partial \tilde{V}^2}{\partial F} + 2\frac{\partial F}{\partial x_\alpha} B^\circ_\alpha\left(\tilde{V}^1, \frac{\partial \tilde{V}^1}{\partial F}\right) - \frac{\partial F}{\partial x_K} L^\circ_{KJ}\frac{\partial F}{\partial x_J}\frac{\partial^2 \tilde{V}^1}{\partial F^2} + A^\circ_\alpha \frac{\partial \tilde{V}^1}{\partial x_\alpha} + \frac{\partial A^\circ_\alpha}{\partial x_\alpha}\tilde{V}^1 = 0 \qquad 109$$

and the experience already acquired indicates that the terms which are not framed are not significant. In the first approximation, we should therefore retain

$$(A^0 + A^1)\frac{\partial \tilde{V}^1}{\partial F} = \mathcal{L}\frac{\partial^2 \tilde{V}^1}{\partial F^2}, \qquad 110$$

provided matrices A^0, A^1 and L are defined as follows:

$$A^0 = \frac{\partial F}{\partial x_\alpha} A^0_\alpha$$
$$A^1 W^1 = \frac{\partial F}{\partial x_\alpha} B^0_\alpha(\tilde{V}^1, W^1) \qquad 111$$
$$\mathcal{L} = \frac{\partial F}{\partial x_K} L^0_{K\beta} \frac{\partial F}{\partial x_J}$$

In cases of interest for applications, the second member of (110) represents a small perturbing term because the norm of matrix L is very small; we shall suppose that such is indeed the case, which leads to the following alternative:

$$a: \begin{cases} (i) & \det(A^0 + A^1) = 0 \\ (ii) & \frac{\partial \tilde{V}^1}{\partial F} \text{ is the eigenvector to the right of } A^0 + A^1, \ \mathcal{L}\frac{\partial^2 \tilde{V}^1}{\partial F^2} \text{ is negligible} \end{cases} \qquad 112$$

b: All the terms of (110) must be retained

which is interpreted by saying that at a given instant and a given point of a given ray, the profile as a whole is obtained by piecing together profiles of short waves in a perfect fluid, joined by shock waves; a little reflection, taking into account the hypothesis of weak perturbation, shows that the profile does not change its structure upon passage of a shock wave, and that consequently one can write from one end of the train to the other,

$$V^1 = f(F, \tau, \mathcal{R})\, W(\tau, \mathcal{R}) \qquad 113$$

This is completed as before, and we thus end up with the evolution equation

/20

$$\frac{\partial f}{\partial \tau} + \rho f \frac{\partial f}{\partial F} + k f = \nu_e \frac{\partial^2 f}{\partial F^2}, \qquad 114$$

provided that we arrange in the definition of F for (97) to take place, and that f (in gas dynamics) represents the perturbation of the component normal to the wave of the velocity vector, so that D_e is interpreted (in gas dynamics) as an effective coefficient of kinematic viscosity whose general expression is given by

$$T \cdot \mathcal{L} \cdot W = \nu_e T A_0 \cdot W, \qquad 115$$

and which more exactly is equal to

$$\nu_e = \frac{1}{2}\left(\frac{\mu_v}{\rho} + \frac{4}{3}\frac{\mu}{\rho} + \frac{(\gamma-1)\kappa}{\rho c_p}\right), \qquad 116$$

in the dynamics of (perfect) gases. Once we are at this point, we enter the well-known domain of application of the Burgers equation to the theory of the supersonic boom.

The literature on the supersonic boom is very abundant, and we shall confine ourselves to the enumeration of certain papers pertaining to the problems touched upon in the preceding pages without trying to present an exhaustive list. The phenomenon of accumulation of perturbations in the vicinity of wave surfaces seems to have been known for a very long time in geometric optics (and of course in quantum mechanics, where it constitutes the essential of the BKW approximation). Its use in linear supersonic aerodynamics has appeared with the Hayes rule of supersonic areas [6]. The analogy between geometric optics and the structure and evolution of discontinuities (or acoustic shock) in linear acoustics was investigated by Keller [7]. The representation (15) was given by Lighthill 13 as a result of the work of Whitham [19, 20] cited below. The discussion of this representation, in the linear case, and that of the representation (113) and (114), in the nonlinear case with dissipation, was given by Guiraud [4] and Goubkin [3] with an indication on how to obtain approximations of higher order. The same representation (113) (114) was given by Lighthill [13] by using a reasoning by analogy, based on the work of Whitham and geometric acoustics as well as on an extensive study of the Burgers equation and its physical implications by Lighthill [13]. The theory itself of the supersonic boom, as applied by engineers, rests on two fundamental articles of Whitham [19, 20], who made a very thorough study of the asymptotic structure of the flow created by the supersonic flight of a fuselage of revolution, far from the latter, and thus discovered the representation (99) (100) in a particular case and recognized therein a case of application of the PLK technique created by Poincaré and applied in aerodynamics by Lighthill [11]. A direct application of this technique to the case of the supersonic boom was presented by Guiraud [4] with, as a result, justified at the cost of laborious estimates with a representation formula, the representation (99) (100), directly specified though it is in its relation with the airplane. A direct and much simpler demonstration on partial differential equations was given by Oswatitsch [14] for nonstationary plane /21
flows. The principal results relative to the Burgers equation are given by Germain [2] with a complete bibliography, and also by Lighthill [13] and Hayes [24] with a discussion of its physical implications. Hayes [7] had previously recognized the role played by equation (16) in problems of nonlinear wave propagation in more than one discussion. A more precise application of the Burgers equation to the estimation of the effects of viscosity in supersonic boom is found in [4]. The description of the effects

of nonlinear convection effects in unidimensional flows had been known for a long time, and a presentation of this description is given in the classical work of Landau and Lifshitz [10], with essentially the demonstration which we gave of the rule of equality of areas for the propagation of shock waves in Fig. 16. This rule was applied by Kristianovich to the prediction of the final asymptotic behavior of the shock wave caused by a very intense explosion; it was rederived by Whitham [20] and is commonly applied by engineers concerned with the prediction of the supersonic boom. Thus, Rijov [15, 17, 18] and Goubkin [3] have indicated how this rule should be applied even in the presence of an inhomogeneous atmosphere and with wind, which amounts to reducing equation (114) to the canonic form (34). An important point for applications is the evaluation of the factor k in equation (114). A little reflection is needed to see that this value should not be affected by nonlinear effects, but this point has now been completely accepted. However, even in linear acoustics, the evaluation of k is not free of traps, and all the authors have not been in immediate agreement on this point. In geometric optics, the evaluation of k results solely from the geometry of the sound rays, and the answer has been known for a long time. In acoustics, in an atmosphere without wind, Whitham discovered the rule as a consequence of his asymptotic study. The difficulties begin with the inhomogeneous atmosphere, and they are compounded if there is a wind and particularly if the atmosphere is time-dependent. Whitham [21] used a rule of conservation of energy in tubes of sound rays, and thus obtained without formal proof, an exact estimate only in the absence of wind. More precise energetic considerations have led Rijov and Schefter [16] to the correct result for a time-independent atmosphere, but the result of Friedman et al [1] is erroneous. The most general case was treated by Guiraud [5] by giving an energetic interpretation essentially equivalent to (70), without, however, noting the underlying symmetrization which we reported after (70) and which makes the result particularly clear. A different energy balance was obtained by Rijov [15], who showed how the acoustic energy dissipated in the internal structure of the shock waves is equivalent, in the first approximation, to the acoustic energy carried by the sound wavelets which disappeared in the shock waves; a similar energy balance but a much more precise one was obtained in a much simpler case by Lighthill [12]. Finally, we do not know of any analysis analogous to that made here starting from formula (102). /22

REFERENCES

1. Friedman, Y.P., Kane, E.J., Sigalla, A. AIAA Journal, Vol. I, #6, p. 1327, 1963.
2. Germain, P., Physics Notes, #119, p. 285, 1960.
3. Goubkin, K.E., P.M.M., Vol. 22, #4, p. 561, 1958.
4. Guiraud, J.P., O.N.E.R.A., N.T., p. 79, 1964; Abstract in ICAS, p. 3, Stockholm, Sept., 1962, Spartan Books, 1965.
5. Guiraud, J.P., Journal de Mecanique, Vol. 4, #2, p. 215, 1965.
6. Hayes, W.D., Caltech Thesis, Techn. Rep. American Aviat., Inc., AL-222, 1947.

7. Hayes, W.D., J.A.S., Vol. 21, #14, p. 721, 1954.
8. Keller, J.B., J. Appl. Physics, Vol. 25, #8, p. 938, 1954.
9. Kristianovich, S.A., P.M.M., Vol. 20, #5, p. 599, 1956.
10. Landau, L.D., Lifshitz, E.M., Fluid Mechanics, pp. 366-377, Pergamon, 1959.
11. Lighthill, M.J., Phil. Mag., Vol. 40, p. 1179, 1949.
12. Lighthill, M.J., Phil. Mag., Vol. 41, p. 1101.
13. Lighthill, M.J., Viscosity effects in sound waves of finite amplitude. Article in Surveys in Mechanics, the G.I. Taylor anniversary volume, edited by Batchelor and Davies, Cambridge University Press, 1956.
14. Oswatitsch, K., Archives of Applied Mechanics, Vol. 14, #3-4, p. 621, 1962.
15. Rijov, O., P.M.M., Vol. 26, #2, p. 316, 1962.
16. Rijov, O., Schefter, G.M., P.M.M., Vol 26, #5, p. 854, 1962.
17. Rijov, O.S., P.M.T., Vol. #2, p. 11, 1961.
18. Rijov, O.S., P.M.T., Vol. #6, p. 37, 1961.
19. Whitham, G.B., Proc. Roy. Soc. A., Vol. 201, p. 89, 1950.
20. Whitham, G.B., Comm. Pure Appl. Math., Vol. 5, #3, p. 338, 1956.
21. Whitham, G.B., J. Fluid Mechanics, Vol. 1, #3, p. 290, 1956.
22. Rao, P., Aero Quarterly, Vol 7, #1, p. 21; #2, p. 135, 1956.
23. Warren, Ch. E., Randall, D.G., The theory of sonic bangs. Article in Progress in Aeronautical Sciences, ed. by Ferri, Pergamon, 1961.
24. Hayes, W.D., The basic theory of gas dynamic discontinuities. Article in Fundamentals of Gas Dynamics. Ed. by Emmons, Princeton University Press, 1956.

Translated for the National Aeronautics and Space Administration by the Translation and Interpretation Division of the Institute of Modern Languages, Inc., under contract NASw-1693.